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# Modelling the Interaction of Two Biological Species in a Polluted Environment

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In this paper a mathematical model is proposed and analysed to study the effect of an environmental pollutant on two interacting biological species. The interaction between the two species is considered to be of three types, namely, competition, cooperation, and prey–predator. In each case criteria for local stability, instability, and global stability of the nonnegative equilibria of the system are obtained. The effect of diffusion on the equilibrium state of the system is also studied. © 2000 Academic Press

## 1. INTRODUCTION

A large amount of pollutants and contaminants released from various industries, motor vehicles, and other man-made projects enter into the environment and affect human population and other biological species seriously. In recent years some investigations have been carried out to study the effect of pollution on a single-species population [2, 4, 6–10, 15]. In particular, Hallam *et al*. [8] studied the effect of a toxicant present in the environment on a single-species population by assuming that its growth rate density decreases linearly with the concentration of toxicant but the corresponding carrying capacity does not depend upon the concentration of toxicant present in the environment. Considering this aspect, Freedman and Shukla [6] studied the effect of toxicant on a single species and on a predator–prey system by taking into account the introduction of toxicant from an external source. Shukla and Dubey [15] studied the simultaneous effect of two toxicants, one being more toxic than the other, on a biological



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species. Dubey [4] proposed a model to study the depletion and conservation of forestry resources in a polluted environment.

We know that species do not exist alone in nature. They interact with other species in their surroundings for their survival. So it is more biologically significant to study two-species systems exposed to a pollutant. In recent decades some investigations have been made to study the system of two biological species [1, 12, 14, 16] in a polluted environment. In particular, Ma Zhien and Hallam [14] studied two-dimensional nonautonomous Lotka–Volterra models by the average method and obtained sufficient conditions for persistence and extinction of the populations. Chattopadhyay [1] studied the effect of toxic substances on a two-species competitive system. He assumed that each of the competing species produces a substance toxic to the other, but only when the other is present. Huaping and Ma Zhien [12] investigated the effects of toxicants on naturally stable two-species communities and derived persistence–extinction criteria for each population. But in modelling the system they assumed that the individuals of the two species have identical organismal toxicant concentration, which need not always be true in nature. Recently, Shukla and Dubey [16] studied the effects of population and pollution on the depletion and conservation of forestry resources. It may be pointed out here that the recycling effect of a toxicant and the effect of diffusion on the stability of the equilibrium state of the system do not appear in the above literature.

In view of the above, in this paper we propose a mathematical model for studying the effect of environmental pollution on two interacting biological species having different organismal pollutant concentrations. Three types of interaction between the two species have been considered, namely, competition, cooperation, and prey–predator. The effect of diffusion on the stability of the system is also studied. In the absence of diffusion our model is more general than that of Huaping and Ma Zhien [12]. In the presence of diffusion our results agree with those in Hastings [11], Shukla and Verma [19], Shukla and Shukla [18], Shukla *et al.* [17], and Freedman and Shukla [5]. The stability theory of the ordinary differential equation is used to analyse the model [13]. In this paper we have also included numerical examples to illustrate the applicability of the results obtained.

## 2. THE MODEL

Consider a polluted environment where two biological species are interacting with each other in a closed region $D$ with smooth boundary $\partial D$. The variables of the model are $x_1 = x_1(x, y, t)$ and $x_2 = x_2(x, y, t)$, the densities of the species 1 and 2 respectively; $T = T(x, y, t)$, the concentration of pollutant present in the environment; $U_1 = U_1(x, y, t)$ and $U_2 = U_2(x, y, t)$, the concentration of pollutant in species 1 and in species 2 respectively at coordinates $(x, y) \in D$ and time $t \geq 0$. In modelling the system we assume that the carrying capacities of the species are constant. Then following Huaping and Ma Zhien [12] and Dubey [4], the Lotka–Volterra model of two species with pollutant effect and diffusion can be written as

$$\begin{aligned}
\frac{\partial x_1}{\partial t} &= r_{10} x_1 - r_{11} x_1 U_1 - a_{11} x_1^2 - a_{12} x_1 x_2 + D_1 \nabla^2 x_1 \\
\frac{\partial x_2}{\partial t} &= r_{20} x_2 - r_{21} x_2 U_2 - a_{21} x_1 x_2 - a_{22} x_2^2 + D_2 \nabla^2 x_2 \\
\frac{\partial T}{\partial t} &= Q_0 - \delta_0 T + \theta_1 \delta_1 U_1 + \theta_2 \delta_2 U_2 - \lambda_1 x_2 T - \lambda_2 x_2 T + D_3 \nabla^2 T \\
\frac{\partial U_1}{\partial t} &= -\delta_1 U_1 + \theta_0 \delta_0 T + \lambda_1 x_1 T + \beta_1 x_1 \\
\frac{\partial U_2}{\partial t} &= -\delta_2 U_2 + \theta_0' \delta_0 T + \lambda_2 x_2 T + \beta_2 x_2
\end{aligned} \tag{2.1}$$

$$0 \leq \theta_0 + \theta_0' \leq 1, \qquad 0 \leq \theta_1, \theta_2 \leq 1.$$

We impose the following initial and boundary conditions on system (2.1)

$$\begin{gathered}
x_1(x, y, 0) = \phi(x, y) \geq 0, \qquad x_2(x, y, 0) = \psi(x, y) \geq 0, \\
T(x, y, 0) = \xi(x, y) \geq 0, \qquad U_1(x, y, 0) = \chi(x, y) \geq 0, \\
U_2(x, y, 0) = \eta(x, y) \geq 0, \qquad (x, y) \in D \\
\frac{\partial x_1}{\partial n} = \frac{\partial x_2}{\partial n} = \frac{\partial T}{\partial n} = \frac{\partial U_1}{\partial n} = \frac{\partial U_2}{\partial n} = 0, \qquad (x, y) \in \partial D, t > 0,
\end{gathered} \tag{2.2}$$

where $n$ is the unit outward normal to $\partial D$.

In model (2.1), $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the Laplacian diffusion operator. $D_i$ $(i = 1, 2, 3)$ are the diffusion rate coefficients of $x_1(x, y, t)$, $x_2(x, y, t)$, and $T(x, y, t)$ respectively in $D$. $r_{i0}$, $r_{i1}$, and $a_{ij}$ $(i, j = 1, 2)$ in the first two equations of model (2.1) are constants. $r_{i0}$ is the intrinsic growth rate of the species $i$ in the absence of pollutant, and $r_{i1}$ the depletion rate coefficient of species $i$ due to organismal pollutant concentration. $a_{12}$ and $a_{21}$ are the interspecific interference coefficients and $a_{11}$, $a_{22}$ are intraspecific interference coefficients of species 1 and 2 respectively. $Q_0$ represents the rate of introduction of pollutant into the environment beyond the initial concentration, which is assumed to be positive or zero. It is assumed that the pollutant in the environment is washed out or broken down with rate $\delta_0$, and fractions $\theta_0$ and $\theta_0'$ of it may again reenter into species 1 and 2 respectively with the uptake of pollutant. $\lambda_1$ and $\lambda_2$ are the depletion rate coefficients of the pollutant in the environment due to its intake by species 1 and 2, respectively. $\delta_1$ and $\delta_2$ are natural depletion rate coefficients of $U_1$ and $U_2$ respectively due to ingestion and depuration of pollutant, and fractions $\theta_1$ and $\theta_2$ of these may again reenter the environment. $\beta_1$ and $\beta_2$ are the net uptake of pollutant from resource by species 1 and 2 respectively.

It is assumed that the parameters $\delta_0$, $\delta_1$, and $\delta_2$ are strictly positive and $\theta_1$, $\theta_2$, $\lambda_1$, $\lambda_2$, $\beta_1$, and $\beta_2$ are nonnegative constants.

The following three cases will be dealt with:

(i) Competition ($r_{10} > 0$, $r_{20} > 0$, $a_{12} > 0$, and $a_{21} > 0$).

(ii) Cooperation ($r_{10} > 0$, $r_{20} > 0$, $a_{12} < 0$, and $a_{21} < 0$).

(iii) Prey–predator ($r_{10} > 0$, $r_{20} < 0$, $a_{12} > 0$, and $a_{21} < 0$), assuming $x_1$ as prey and $x_2$ as predator.

## 3. COMPETITION MODEL

We first analyse model (2.1) without diffusion (i.e., $D_1 = D_2 = D_3 = 0$).

### 3.1. *No Diffusion*

In this case model (2.1) has four nonnegative equilibria, viz,

$$E_0\left(0, 0, \frac{Q_0}{\delta_0(1 - \theta_0\theta_1 - \theta_0'\theta_2)}, \frac{\theta_0 Q_0}{\delta_1(1 - \theta_0\theta_1 - \theta_0'\theta_2)}, \frac{\theta_0' Q_0}{\delta_2(1 - \theta_0\theta_1 - \theta_0'\theta_2)}\right),$$

$$\tilde{E}\left(\tilde{x}_1, 0, \tilde{T}, \tilde{U}_1, \tilde{U}_2\right), \qquad \hat{E}\left(0, \hat{x}_2, \hat{T}, \hat{U}_1, \hat{U}_2\right), \qquad \text{and}$$

$$\overline{E}\left(\bar{x}_1, \bar{x}_2, \overline{T}, \overline{U}_1, \overline{U}_2\right).$$

The equilibrium $E_0$ exists if

$$1 - \theta_0\theta_1 - \theta_0'\theta_2 > 0. \tag{3.1}$$

We shall show the existence of three other equilibria as follows:

*Existence of* $\tilde{E}(\tilde{x}_1, 0, \tilde{T}, \tilde{U}_1, \tilde{U}_2)$. $\tilde{x}_1$, $\tilde{T}$, $\tilde{U}_1$, and $\tilde{U}_2$ are the positive solutions of the following algebraic equations

$$\begin{aligned}
a_{11}x_1 &= r_{10} - r_{11}U_1,\\
\delta_0 T + \lambda_1 x_1 T &= Q_0 + \theta_1\,\delta_1 U_1 + \theta_2\,\delta_2 U_2,\\
\delta_1 U_1 &= \theta_0\,\delta_0 T + \lambda_1 x_1 T + \beta_1 x_1,\\
\delta_2 U_2 &= \theta_0'\,\delta_0 T.
\end{aligned}$$

A little algebraic manipulation yields

$$a_{11}x_1 = r_{10} - r_{11}g(x_1),$$

$$T = \frac{Q_0 + \theta_1\,\beta_1 x_1}{\delta_0(1 - \theta_0\theta_1 - \theta_0'\theta_2) + \lambda_1(1 - \theta_1)x_1} = f(x_1),$$

say,

$$U_1 = \frac{1}{\delta_1}\{\theta_0\,\delta_0 f(x_1) + \lambda_1 x_1 f(x_1) + \beta_1 x_1\} = g(x_1),$$

say,

$$U_2 = \frac{1}{\delta_2}\theta_0'\,\delta_0 f(x_1) = h(x_1),$$

say. Taking

$$F(x_1) = a_{11}x_1 - r_{10} + r_{11}g(x_1)$$

we note that $F(0) < 0$ if

$$r_{11}\theta_0 Q_0 < r_{10}\delta_1(1 - \theta_0\theta_1 - \theta_0'\theta_2) \tag{3.2}$$

and $F(r_{10}/a_{11}) > 0$, showing the existence of $\tilde{x}_1$ in the interval $0 < \tilde{x}_1 < r_{10}/a_{11}$. For $\tilde{x}_1$ to be unique the following condition must be satisfied at $\tilde{E}$;

$$Q_0\lambda_1(1 - \theta_1) < \theta_1\,\beta_1\delta_0(1 - \theta_0\theta_1 - \theta_0'\theta_2). \tag{3.3}$$

Thus from the above analysis we note that the equilibrium $\tilde{E}$ exists under conditions (3.2) and (3.3).

*Existence of* $\hat{E}(0, \hat{x}_2, \hat{T}, \hat{U}_1, \hat{U}_2)$. As in the existence of $\tilde{E}$, it can be seen that the equilibrium $\hat{E}$ exists if the following inequalities hold

$$r_{21}\theta_0' Q_0 < r_{20}\delta_2(1 - \theta_0\theta_1 - \theta_0'\theta_2), \tag{3.4}$$

$$Q_0\lambda_2(1 - \theta_2) < \theta_2\beta_2\delta_0(1 - \theta_0\theta_1 - \theta_0'\theta_2). \tag{3.5}$$

*Existence of* $\overline{E}(\bar{x}_1, \bar{x}_2, \overline{T}, \overline{U}_1, \overline{U}_2)$. Here, $\bar{x}_1$, $\bar{x}_2$, $\overline{T}$, $\overline{U}_1$, and $\overline{U}_2$ are the positive solutions of the system of algebraic equations

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + r_{11}g(x_1, x_2) &= r_{10},\\
a_{21}x_1 + a_{22}x_2 + r_{21}h(x_1, x_2) &= r_{20},\\
T &= f(x_1, x_2),\\
U_1 &= g(x_1, x_2),\\
U_2 &= h(x_1, x_2),
\end{aligned}$$

where

$$f(x_1, x_2) = \frac{Q_0 + \theta_1\beta_1x_1 + \theta_2\beta_2x_2}{\delta_0(1 - \theta_0\theta_1 - \theta_0'\theta_2) + \lambda_1(1 - \theta_1)x_1 + \lambda_2(1 - \theta_2)x_2},$$

$$g(x_1, x_2) = \frac{1}{\delta_1}\{\theta_0\,\delta_0 f(x_1, x_2) + \lambda_1x_1f(x_1, x_2) + \beta_1x_1\},$$

$$h(x_1, x_2) = \frac{1}{\delta_2}\{\theta_0'\,\delta_0 f(x_1, x_2) + \lambda_2x_2f(x_1, x_2) + \beta_2x_2\}.$$

It can be checked easily that $\overline{E}$ exists if in addition to conditions (3.2) and (3.4), the conditions

$$\frac{-b + (b^2 - 4ac)^{1/2}}{2a} > \frac{-b' + (b'^2 - 4a'c')^{1/2}}{2a'}, \tag{3.6a}$$

$$\frac{-B + (B^2 - 4AC)^{1/2}}{2A} > \frac{-B' + (B'^2 - 4A'C')^{1/2}}{2A'}, \tag{3.6b}$$

$$\frac{a_{12} + r_{11}(\partial g/\partial x_2)}{a_{11} + r_{11}(\partial g/\partial x_1)} > 0, \tag{3.7a}$$

$$\frac{a_{22} + r_{21}(\partial h/\partial x_2)}{a_{21} + r_{21}(\partial h/\partial x_1)} > 0, \tag{3.7b}$$

hold, where

$$
\begin{aligned}
a &= \lambda_1\{a_{11}\delta_1(1-\theta_1) + r_{11}\beta_1\},\\
b &= a_{11}\delta_0\delta_1(1-\theta_0\theta_1-\theta_0'\theta_2) + r_{11}\delta_0\beta_1(1-\theta_0'\theta_2) + r_{11}\lambda_1 Q_0\\
&\quad - \lambda_1\delta_1 r_{10}(1-\theta_1),\\
c &= r_{11}\theta_0\delta_0 Q_0 - \delta_0\delta_1 r_{10}(1-\theta_0\theta_1-\theta_0'\theta_2),\\
a' &= \lambda_1 a_{21}\delta_2(1-\theta_1),\\
b' &= a_{21}\delta_0\delta_2(1-\theta_0\theta_1-\theta_0'\theta_2) + r_{21}\theta_0'\delta_0\theta_1\beta_1 - \lambda_1\delta_2 r_{20}(1-\theta_1),\\
c' &= r_{21}\theta_0'\delta_0 Q_0 - \delta_0\delta_2 r_{20}(1-\theta_0\theta_1-\theta_0'\theta_2),\\
A &= \lambda_2\{a_{22}\delta_2(1-\theta_2) + r_{21}\beta_2\},\\
B &= a_{22}\delta_0\delta_2(1-\theta_0\theta_1-\theta_0'\theta_2) + r_{21}\delta_0\beta_2(1-\theta_0\theta_1) + r_{21}\lambda_2 Q_0\\
&\quad - \lambda_2\delta_2 r_{20}(1-\theta_2),\\
C &= r_{21}\theta_0'\delta_0 Q_0 - \delta_0\delta_2 r_{20}(1-\theta_0\theta_1-\theta_0'\theta_2),\\
A' &= \lambda_2 a_{12}\delta_1(1-\theta_2),\\
B' &= a_{12}\delta_0\delta_1(1-\theta_0\theta_1-\theta_0'\theta_2) + r_{11}\theta_0\delta_0\theta_2\beta_2 - \lambda_2\delta_1 r_{10}(1-\theta_2),\\
C' &= r_{11}\theta_0\delta_0 Q_0 - \delta_0\delta_1 r_{10}(1-\theta_0\theta_1-\theta_0'\theta_2).
\end{aligned}
$$

It may be noted here that $\overline{E}$ exists even when the inequalities (3.6a) and (3.6b) are reversed.

To study the local stability behavior of the equilibria, we first compute the variational matrices corresponding to each equilibrium point. From these matrices we conclude the following:

$E_0$ is a saddle point with an unstable manifold locally in the $x_1 - x_2$ plane and a stable manifold locally in the $T - U_1 - U_2$ space. $\tilde{E}$ and $\hat{E}$ are locally unstable in the $x_2$ and $x_1$ directions respectively.

In the following theorem we have shown that $\overline{E}$ is locally asymptotically stable.

THEOREM 3.1. *Let the inequalities*

$$(a_{12}+a_{21})^2 < \tfrac{4}{9}a_{11}a_{22} \tag{3.8a}$$

$$\{c_1'\theta_1\delta_1 + c_2'(\theta_0\delta_0+\lambda_1\bar{x}_1)\}^2 < \tfrac{1}{2}c_1'c_2'\delta_1(\delta_0+\lambda_1\bar{x}_1+\lambda_2\bar{x}_2) \tag{3.8b}$$

$$\{c_1'\theta_2\delta_2 + c_3'(\theta_0'\delta_0+\lambda_2\bar{x}_2)\}^2 < \tfrac{1}{2}c_1'c_3'\delta_2(\delta_0+\lambda_1\bar{x}_1+\lambda_2\bar{x}_2) \tag{3.8c}$$

*hold*, *where*

$$c_1' = \min\left\{\frac{1}{4}\frac{a_{11}}{\lambda_1^2}\frac{\delta_0 + \lambda_1\bar{x}_1 + \lambda_2\bar{x}_2}{\overline{T}^2}, \frac{1}{4}\frac{a_{22}}{\lambda_2^2}\frac{\delta_0 + \lambda_1\bar{x}_1 + \lambda_2\bar{x}_2}{\overline{T}^2}\right\},$$

$$c_2' = \frac{r_{11}}{\lambda_1\overline{T} + \beta_1},$$

$$c_3' = \frac{r_{21}}{\lambda_2\overline{T} + \beta_2}.$$

*Then $\overline{E}$ is locally asymptotically stable.*

*Proof.* Consider the following positive definite function in the linearized form of system (2.1) with no diffusion,

$$V(x_1, x_2, T, U_1, U_2) = \frac{(x_1 - \bar{x}_1)^2}{2\bar{x}_1} + \frac{(x_2 - \bar{x}_2)^2}{2\bar{x}_2} + \frac{c_1'}{2}(T - \overline{T})^2 + \frac{c_2'}{2}(U_1 - \overline{U}_1)^2 + \frac{c_3'}{2}(U_2 - \overline{U}_2)^2.$$

It can be seen easily that the derivative of $V$ with respect to $t$ along the solution of model (2.1) with no diffusion is negative definite under conditions (3.8), proving the theorem.

To show that $\overline{E}$ is globally asymptotically stable, we need the following lemma, which establishes a region of attraction for system (2.1). The proof of this lemma is easy and hence is omitted.

LEMMA 3.1. *The set*

$$\Omega_1 = \left\{(x_1, x_2, T, U_1, U_2) : 0 \le x_1 \le \frac{r_{10}}{a_{11}}, 0 \le x_2 \le \frac{r_{20}}{a_{22}}, 0 \le T + U_1 + U_2 \le L_1\right\}$$

*attracts all solutions initiating in the positive orthant*, *where*

$$L_1 = \frac{1}{\delta}\left\{Q_0 + \beta_1\frac{r_{10}}{a_{11}} + \beta_2\frac{r_{20}}{a_{22}}\right\},$$

$$\delta = \min\{\delta_0(1 - \theta_0 - \theta_0'), \delta_1(1 - \theta_1), \delta_2(1 - \theta_2)\}.$$

In the following theorem the global stability of $\overline{E}$ is studied.

THEOREM 3.2. *Let the inequalities*

$$(a_{12} + a_{21})^2 < \tfrac{4}{9} a_{11} a_{22} \tag{3.9a}$$

$$\{c_1 \theta_1 \delta_1 + c_2(\theta_0 \delta_0 + \lambda_1 \bar{x}_1)\}^2 < \tfrac{1}{2} c_1 c_2 \delta_1(\delta_0 + \lambda_1 \bar{x}_1 + \lambda_2 \bar{x}_2) \tag{3.9b}$$

$$\{c_1 \theta_2 \delta_2 + c_3(\theta_0' \delta_0 + \lambda_2 \bar{x}_2)\}^2 < \tfrac{1}{2} c_1 c_3 \delta_2(\delta_0 + \lambda_1 \bar{x}_1 + \lambda_2 \bar{x}_2) \tag{3.9c}$$

*hold, where*

$$c_1 = \min\left\{ \frac{1}{4} \frac{a_{11}}{\lambda_1^2} \frac{\delta_0 + \lambda_1 \bar{x}_1 + \lambda_2 \bar{x}_2}{L_1^2}, \frac{1}{4} \frac{a_{22}}{\lambda_2^2} \frac{\delta_0 + \lambda_1 \bar{x}_1 + \lambda_2 \bar{x}_2}{L_1^2} \right\},$$

$$c_2 = \frac{r_{11}}{\lambda_1 L_1 + \beta_1},$$

$$c_3 = \frac{r_{21}}{\lambda_2 L_1 + \beta_2}.$$

*Then $\bar{E}$ is globally asymptotically stable with respect to all solutions initiating in the interior of the positive orthant.*

*Proof.* Consider the positive definite function around $\bar{E}$,

$$W(x_1, x_2, T, U_1, U_2) = x_1 - \bar{x}_1 - \bar{x}_1 \ln\left(\frac{x_1}{\bar{x}_1}\right) + x_2 - \bar{x}_2 - \bar{x}_2 \ln\left(\frac{x_2}{\bar{x}_2}\right) + \frac{c_1}{2}(T - \bar{T})^2 + \frac{c_2}{2}(U_1 - \bar{U}_1)^2 + \frac{c_3}{2}(U_2 - \bar{U}_2)^2. \tag{3.10}$$

Differentiating $W$ with respect to $t$ along the solutions of system (2.1) without diffusion, we get

$$\begin{aligned} \frac{dW}{dt} &= (x_1 - \bar{x}_1)[r_{10} - r_{11}U_1 - a_{11}x_1 - a_{12}x_2] \\ &\quad + (x_2 - \bar{x}_2)[r_{20} - r_{21}U_2 - a_{21}x_1 - a_{22}x_2] \\ &\quad + c_1(T - \bar{T})[Q_0 - \delta_0 T + \theta_1 \delta_1 U_1 + \theta_2 \delta_2 U_2 - \lambda_1 x_1 T - \lambda_2 x_2 T] \\ &\quad + c_2(U_1 - \bar{U}_1)[-\delta_1 U_1 + \theta_0 \delta_0 T + \lambda_1 x_1 T + \beta_1 x_1] \\ &\quad + c_3(U_2 - \bar{U}_2)[-\delta_2 U_2 + \theta_0' \delta_0 T + \lambda_2 x_2 T + \beta_2 x_2]. \end{aligned} \tag{3.11}$$

After some algebraic manipulations, Eq. (3.11) can be written as

$$\begin{aligned}\frac{dW}{dt} = &-\frac{1}{2}A_{11}(x_1-\bar{x}_1)^2 + A_{12}(x_1-\bar{x}_1)(x_2-\bar{x}_2) - \frac{1}{2}A_{22}(x_2-\bar{x}_2)^2\\
&-\frac{1}{2}A_{11}(x_1-\bar{x}_1)^2 + A_{13}(x_1-\bar{x}_1)(T-\overline{T}) - \frac{1}{2}A_{33}(T-\overline{T})^2\\
&-\frac{1}{2}A_{11}(x_1-\bar{x}_1)^2 + A_{14}(x_1-\bar{x}_1)\left(U_1-\overline{U}_1\right) - \frac{1}{2}A_{44}\left(U_1-\overline{U}_1\right)^2\\
&-\frac{1}{2}A_{22}(x_2-\bar{x}_2)^2 + A_{23}(x_2-\bar{x}_2)(T-\overline{T}) - \frac{1}{2}A_{33}(T-\overline{T})^2\\
&-\frac{1}{2}A_{22}(x_2-\bar{x}_2)^2 + A_{25}(x_2-\bar{x}_2)\left(U_2-\overline{U}_2\right) - \frac{1}{2}A_{55}\left(U_2-\overline{U}_2\right)^2\\
&-\frac{1}{2}A_{33}(T-\overline{T})^2 + A_{34}(T-\overline{T})\left(U_1-\overline{U}_1\right) - \frac{1}{2}A_{44}\left(U_1-\overline{U}_1\right)^2\\
&-\frac{1}{2}A_{33}(T-\overline{T})^2 + A_{35}(T-\overline{T})\left(U_2-\overline{U}_2\right) - \frac{1}{2}A_{55}\left(U_2-\overline{U}_2\right)^2,\end{aligned}$$

where

$$A_{11} = \tfrac{2}{3}a_{11}, \quad A_{22} = \tfrac{2}{3}a_{22}, \quad A_{33} = \tfrac{1}{2}c_1\left(\delta_0 + \lambda_1\bar{x}_1 + \lambda_2\bar{x}_2\right),$$

$$A_{44} = c_2\delta_1, \quad A_{55} = c_3\delta_2, \quad A_{12} = -(a_{12}+a_{21}), \quad A_{13} = -c_1\lambda_1 T,$$

$$A_{14} = c_2(\lambda_1 T + \beta_1) - r_{11}, \quad A_{23} = -c_1\lambda_2 T,$$

$$A_{25} = c_3(\lambda_2 T + \beta_2) - r_{21}, \quad A_{34} = c_1\theta_1\delta_1 + c_2(\theta_0\delta_0 + \lambda_1\bar{x}_1),$$

$$A_{35} = c_1\theta_2\delta_2 + c_3\left(\theta_0'\delta_0 + \lambda_2\bar{x}_2\right).$$

Sufficient conditions for $dW/dt$ to be negative definite are that the following inequalities hold

$$A_{12}^2 < A_{11}A_{22}, \tag{3.12a}$$

$$A_{13}^2 < A_{11}A_{33}, \tag{3.12b}$$

$$A_{14}^2 < A_{11}A_{44}, \tag{3.12c}$$

$$A_{23}^2 < A_{22}A_{33}, \tag{3.12d}$$

$$A_{25}^2 < A_{22}A_{55}, \tag{3.12e}$$

$$A_{34}^2 < A_{33}A_{44}, \tag{3.12f}$$

$$A_{35}^2 < A_{33}A_{55}. \tag{3.12g}$$

Under the suitable choice of constants $c_1, c_2, c_3$ as in Theorem 3.2, we note that inequalities (3.12b)–(3.12e) are automatically satisfied and (3.9a) $\Rightarrow$ (3.12a), (3.9b) $\Rightarrow$ (3.12f), and (3.9c) $\Rightarrow$ (3.12g). Thus $W$ is a Liapunov function with respect to $\overline{E}$, whose domain contains the region $\Omega_1$, proving the theorem.

*Remark* 3.1. In the case of instantaneous introduction of a pollutant (i.e., $Q_0 = 0$) into the environment, it can be verified that there are four nonnegative equilibria, namely $E_0(0,0,0,0,0)$, $\tilde{E}(\tilde{x}_1, 0, \tilde{T}, \tilde{U}_1, \tilde{U}_2)$, $\hat{E}(0, \hat{x}_2, \hat{T}, \hat{U}_1, \hat{U}_2)$, and $\overline{E}(\bar{x}_1, \bar{x}_2, \overline{T}, \overline{U}_1, \overline{U}_2)$. $E_0$ obviously exists and the existence of the remaining three equilibria can be seen in a fashion similar to that discussed earlier. Further, the stability behavior of the equilibria is similar to the corresponding equilibria as given in the case of the constant introduction of pollutant into the environment. It has been noted here that equilibrium levels of the competing species in the constant introduction of pollutant into the environment is lower than the case of instantaneous introduction, keeping other parameters and functions the same in the model.

### 3.2. *Model with Diffusion*

In this section we consider the complete model (2.1)–(2.2) and we state the main results of this section in the form of the following theorem.

THEOREM 3.3. (i) *If the equilibrium $\overline{E}$ of the system without diffusion is globally asymptotically stable, then the corresponding uniform steady state of the initial-boundary value problems* (2.1)–(2.2) *is also globally asymptotically stable.*

(ii) *If the equilibrium $\overline{E}$ of the system without diffusion is unstable, even the uniform steady state of the initial-boundary value problems* (2.1)–(2.2) *can be made stable by increasing diffusion coefficients appropriately.*

*Proof.* Let us consider the positive definite function

$$U(x_1(t), x_2(t), T(t), U_1(t), U_2(t)) = \iint_D W(x_1, x_2, T, U_1, U_2)\, dA, \tag{3.13}$$

where $W$ is defined by Eq. (3.10).

We have

$$\frac{dU}{dt} = \iint_D \left[ \frac{\partial W}{\partial x_1}\frac{\partial x_1}{\partial t} + \frac{\partial W}{\partial x_2}\frac{\partial x_2}{\partial t} + \frac{\partial W}{\partial T}\frac{\partial T}{\partial t} + \frac{\partial W}{\partial U_1}\frac{\partial U_1}{\partial t} + \frac{\partial W}{\partial U_2}\frac{\partial U_2}{\partial t} \right] dA$$

$$= \iint_D \dot{W}\, dA + \iint_D \left[ D_1 \frac{\partial W}{\partial x_1}\nabla^2 x_1 + D_2 \frac{\partial W}{\partial x_2}\nabla^2 x_2 + D_3 \frac{\partial W}{\partial T}\nabla^2 T \right] dA$$

$$= I_1 + I_2, \tag{3.14}$$

where

$$I_1 = \iint_D \dot{W}\, dA, \qquad \text{and}$$

$$I_2 = \iint_D \left[ D_1 \frac{\partial W}{\partial x_1}\nabla^2 x_1 + D_2 \frac{\partial W}{\partial x_2}\nabla^2 x_2 + D_3 \frac{\partial W}{\partial T}\nabla^2 T \right] dA.$$

We note the following properties of $W$, namely,

$$\left.\frac{\partial W}{\partial x_1}\right|_{\partial D} = \left.\frac{\partial W}{\partial x_2}\right|_{\partial D} = \left.\frac{\partial W}{\partial T}\right|_{\partial D} = \left.\frac{\partial W}{\partial U_1}\right|_{\partial D} = \left.\frac{\partial W}{\partial U}\right|_{\partial D} = 0,$$

and for all points of $D$,

$$\frac{\partial^2 W}{\partial x_1\,\partial x_2} = \frac{\partial^2 W}{\partial x_1\,\partial T} = \frac{\partial^2 W}{\partial x_1\,\partial U_1} = \frac{\partial^2 W}{\partial x_1\,\partial U_2} = \frac{\partial^2 W}{\partial x_2\,\partial T} = \frac{\partial^2 W}{\partial x_2\,\partial U_1}$$
$$= \frac{\partial^2 W}{\partial x_2\,\partial U_2} = \frac{\partial^2 W}{\partial T\,\partial U_1} = \frac{\partial^2 W}{\partial T\,\partial U_2} = \frac{\partial^2 W}{\partial U_1\,\partial U_2} = 0$$

and

$$\frac{\partial^2 W}{\partial x_1^2} > 0, \quad \frac{\partial^2 W}{\partial x_2^2} > 0, \quad \frac{\partial^2 W}{\partial T^2} > 0, \quad \frac{\partial^2 W}{\partial U_1^2} > 0, \quad \frac{\partial^2 W}{\partial U_2^2} > 0.$$

We now consider $I_2$ and determine the sign of each term. We utilize the formula known as Green's first identity in the plane

$$\iint_D F\,\nabla^2 G\, dA = \int_{\partial D} F \frac{\partial G}{\partial n}\, ds - \iint_D (\nabla F . \nabla G)\, dA,$$

where $\partial G/\partial n$ is the directional derivative in the direction of the unit outward normal to $\partial D$ and $s$ is the arc length.

Then with $F = \partial W/\partial x_1$ and $G = x_1$, we get

$$\iint_D \frac{\partial W}{\partial x_1} \nabla^2 x_1 \, dA = \int_{\partial D} \frac{\partial W}{\partial x_1} \frac{\partial x_1}{\partial n} \, ds - \iint_D \left[ \nabla \left( \frac{\partial W}{\partial x_1} \right) . \nabla x_1 \right] dA$$

$$= - \iint_D \left[ \nabla \left( \frac{\partial W}{\partial x_1} \right) . \nabla x_1 \right] dA,$$

since $\partial x_1/\partial n = 0$.

Now,

$$\nabla \left( \frac{\partial W}{\partial x_1} \right) = \frac{\partial^2 W}{\partial x_1^2} \frac{\partial x_1}{\partial x} \hat{i} + \frac{\partial^2 W}{\partial x_1^2} \frac{\partial x_1}{\partial y} \hat{j}.$$

Hence,

$$\iint_D \frac{\partial W}{\partial x_1} \nabla^2 x_1 \, dA = - \iint_D \left( \frac{\partial^2 W}{\partial x_1^2} \right) \left[ \left( \frac{\partial x_1}{\partial x} \right)^2 + \left( \frac{\partial x_1}{\partial y} \right)^2 \right] dA \le 0. \quad (3.15a)$$

Similarly,

$$\iint_D \frac{\partial W}{\partial x_2} \nabla^2 x_2 \, dA \le 0 \qquad \text{and} \qquad \iint_D \frac{\partial W}{\partial T} \nabla^2 T \, dA \le 0. \quad (3.15b)$$

i.e., $I_2 \le 0$.

Thus we note that if $I_1 \le 0$, i.e., if $\overline{E}$ is globally asymptotically stable in the absence of diffusion, then the uniform steady state of the initial-boundary value problems (2.1)–(2.2) also must be globally asymptotically stable. This proves the first part of the theorem.

We further note that if $\dot{W} > 0$, i.e., $I_1 > 0$, then $\overline{E}$ will be unstable in the absence of diffusion. However, Eqs. (3.14) and (3.15) show that by making the diffusion coefficients $D_i$ sufficiently large, $\dot{U}$ can be made negative even if $I_1 > 0$. This proves the second part of the theorem.

Now we shall prove Theorem 3.3 for a rectangular region. Let us consider $D$ to be a rectangular region given by

$$D = \{ (x, y) : 0 \le x \le a, 0 \le y \le b \}$$

In this case $I_2$ can be written as

$$I_2 = -D_1 \iint_D \left( \frac{\partial^2 W}{\partial x_1^2} \right) \left[ \left( \frac{\partial x_1}{\partial x} \right)^2 + \left( \frac{\partial x_1}{\partial y} \right)^2 \right] dA + \text{negative terms}.$$

However,

$$\frac{\partial^2 W}{\partial x_1^2} = \frac{\bar{x}_1}{x_1^2}$$

So,

$$\begin{aligned}
I_2 &= -D_1\left(\frac{\bar{x}_1}{x_1^2}\right)\iint_D\left[\left(\frac{\partial x_1}{\partial x}\right)^2 + \left(\frac{\partial x_1}{\partial y}\right)^2\right] dA + \text{negative terms} \\
&\leq -D_1\left(\frac{a_{11}^2\bar{x}_1}{r_{10}^2}\right)\iint_D\left[\left(\frac{\partial x_1}{\partial x}\right)^2 + \left(\frac{\partial x_1}{\partial y}\right)^2\right] dA.
\end{aligned}$$

Now,

$$\begin{aligned}
\iint_D\left(\frac{\partial x_1}{\partial x}\right)^2 dA &= \iint_D\left(\frac{\partial(x_1 - \bar{x}_1)}{\partial x}\right)^2 dA \\
&= \int_0^b\int_0^a\left(\frac{\partial(x_1 - \bar{x}_1)}{\partial x}\right)^2 dx\,dy.
\end{aligned}$$

Let $z = x/a$, then

$$\iint_D\left(\frac{\partial x_1}{\partial x}\right)^2 dA = \frac{1}{a}\int_0^b\int_0^1\left(\frac{\partial(x_1 - \bar{x}_1)}{\partial z}\right)^2 dz\,dy.$$

Now, utilizing the well-known inequality [3]

$$\int_0^1\left(\frac{\partial x_1}{\partial x}\right)^2 dx \geq \pi^2\int_0^1 x_1^2\,dx,$$

we get

$$\begin{aligned}
\iint_D\left(\frac{\partial x_1}{\partial x}\right)^2 dA &\geq \frac{\pi^2}{a}\int_0^b\int_0^1(x_1 - \bar{x}_1)^2\,dz\,dy \\
&= \frac{\pi^2}{a^2}\int_0^b\int_0^a(x_1 - \bar{x}_1)^2\,dx\,dy \\
&= \frac{\pi^2}{a^2}\iint_D(x_1 - \bar{x}_1)^2\,dA.
\end{aligned}$$

Similarly,

$$\iint_D \left( \frac{\partial x_1}{\partial y} \right)^2 dA \geq \frac{\pi^2}{b^2} \iint_D (x_1 - \bar{x}_1)^2 \, dA,$$

hence,

$$I_2 \leq -D_1 \frac{a_{11}^2 \bar{x}_1}{r_{10}^2} \iint_D \pi^2 \left( \frac{1}{a^2} + \frac{1}{a^2} \right) (x_1 - \bar{x}_1)^2 \, dA$$
$$= -\frac{D_1 a_{11}^2 \bar{x}_1 \pi^2 (a^2 + b^2)}{r_{10}^2 a^2 b^2} \iint_D (x_1 - \bar{x}_1)^2 \, dA,$$

which shows that $I_2 \leq 0$.

Thus for a given rectangular region, by increasing diffusion coefficients sufficiently large, an unstable steady state in the absence of diffusion can be made stable. This theorem implies that in the presence of diffusion the competing species converge towards their respective carrying capacities faster than in the case of no diffusion.

## 4. COOPERATION MODEL

In this case we have $r_{10} > 0$, $r_{20} > 0$, $a_{12} < 0$, and $a_{21} < 0$. There exist four nonnegative equilibria, namely,

$$E_0 \left( 0, 0, \frac{Q_0}{\delta_0 (1 - \theta_0 \theta_1 - \theta_0' \theta_2)}, \frac{\theta_0 Q_0}{\delta_1 (1 - \theta_0 \theta_1 - \theta_0' \theta_2)}, \frac{\theta_0' Q_0}{\delta_2 (1 - \theta_0 \theta_1 - \theta_0' \theta_2)} \right),$$

$$\tilde{E}_c \left( \tilde{x}_{1c}, 0, \tilde{T}_c, \tilde{U}_{1c}, \tilde{U}_{2c} \right),$$

$$\hat{E}_c \left( 0, \hat{x}_{2c}, \hat{T}_c, \hat{U}_{1c}, \hat{U}_{2c} \right),$$

and

$$\overline{E}_c \left( \bar{x}_{1c}, \bar{x}_{2c}, \overline{T}_c, \overline{U}_{1c}, \overline{U}_{2c} \right).$$

$E_0$ exists if $1 - \theta_0 \theta_1 - \theta_0' \theta_2 > 0$. Existence of $\tilde{E}_c$, $\hat{E}_c$, and $\overline{E}_c$ can be checked as was done in the competition model in Section 3.

The local stability behaviors of $E_0$, $\tilde{E}_c$, and $\hat{E}_c$ are similar to the corresponding equilibria of Section 3.

The following theorem shows the local stability character of $\overline{E}_c$, the proof of which is similar to Theorem 3.1 and hence is omitted.

THEOREM 4.1. *Let the inequalities*

$$(a_{12} + a_{21})^2 < \tfrac{4}{9} a_{11} a_{22} \tag{4.1a}$$

$$\{k_1' \theta_1 \delta_1 + k_2'(\theta_0 \delta_0 + \lambda_1 \bar{x}_{1c})\}^2 < \tfrac{1}{2} k_1' k_2' \delta_1(\delta_0 + \lambda_1 \bar{x}_{1c} + \lambda_2 \bar{x}_{2c}) \tag{4.1b}$$

$$\{k_1' \theta_2 \delta_2 + k_3'(\theta_0' \delta_0 + \lambda_2 \bar{x}_{2c})\}^2 < \tfrac{1}{2} k_1' k_3' \delta_2(\delta_0 + \lambda_1 \bar{x}_{1c} + \lambda_2 \bar{x}_{2c}) \tag{4.1c}$$

*hold, where*

$$\begin{aligned} k_1' &= \min\left\{ \frac{1}{4} \frac{a_{11}}{\lambda_1^2} \frac{\delta_0 + \lambda_1 \bar{x}_{1c} + \lambda_2 \bar{x}_{2c}}{\overline{T}_c^2}, \frac{1}{4} \frac{a_{22}}{\lambda_2^2} \frac{\delta_0 + \lambda_1 \bar{x}_{1c} + \lambda_2 \bar{x}_{2c}}{\overline{T}_c^2} \right\}, \\ k_2' &= \frac{r_{11}}{\lambda_1 \overline{T}_c + \beta_1}, \\ k_3' &= \frac{r_{21}}{\lambda_2 \overline{T}_c + \beta_2}, \end{aligned}$$

*then $\overline{E}_c$ is locally asymptotically stable.*

In order to show the global stability of $\overline{E}_c$, we need the following lemma whose proof is easy and hence is omitted.

LEMMA 4.1. *The set*

$$\begin{aligned} \Omega_2 = \{(x_1, x_2, T, U_1, U_2) : 0 \le x_1 \le x_{1\infty} < \infty, 0 \le x_2 \le x_{2\infty} < \infty, \\ 0 \le T + U_1 + U_2 \le L_2\} \end{aligned}$$

*attracts all the solutions initiating in the interior of the positive orthant, where*

$$\begin{aligned} L_2 &= \frac{1}{\delta}(Q_0 + \beta_1 x_{1\infty} + \beta_2 x_{2\infty}), \\ \delta &= \min\{\delta_0(1 - \theta_0 - \theta_0'), \delta_1(1 - \theta_1), \delta_2(1 - \theta_2)\}. \end{aligned}$$

The following theorem shows the global stability of $\overline{E}_c$ whose proof is similar to that of Theorem 3.2 and hence is omitted.

THEOREM 4.2. *Let the inequalities*

$$(a_{12} + a_{21})^2 < \tfrac{4}{9} a_{11} a_{22} \tag{4.2a}$$

$$\{k_1 \theta_1 \delta_1 + k_2(\theta_0 \delta_0 + \lambda_1 \bar{x}_{1c})\}^2 < \tfrac{1}{2} k_1 k_2 \delta_1(\delta_0 + \lambda_1 \bar{x}_{1c} + \lambda_2 \bar{x}_{2c}) \tag{4.2b}$$

$$\{k_1 \theta_2 \delta_2 + k_3(\theta'_0 \delta_0 + \lambda_2 \bar{x}_{2c})\}^2 < \tfrac{1}{2} k_1 k_3 \delta_2(\delta_0 + \lambda_1 \bar{x}_{1c} + \lambda_2 \bar{x}_{2c}) \tag{4.2c}$$

*hold*, *where*

$$k_1 = \min\left\{\frac{1}{4}\frac{a_{11}}{\lambda_1^2}\frac{\delta_0 + \lambda_1 \bar{x}_{1c} + \lambda_2 \bar{x}_{2c}}{L_2^2}, \frac{1}{4}\frac{a_{22}}{\lambda_2^2}\frac{\delta_0 + \lambda_1 \bar{x}_{1c} + \lambda_2 \bar{x}_{2c}}{L_2^2}\right\},$$

$$k_2 = \frac{r_{11}}{\lambda_1 L_2 + \beta_1},$$

$$k_3 = \frac{r_{21}}{\lambda_2 L_2 + \beta_2},$$

*then* $\overline{E}_c$ *is globally asymptotically stable with respect to all solutions initiating in the interior of the positive orthant.*

It may be noted here that conditions in Theorem 4.1 are similar to Theorem 3.1, and conditions in Theorem 4.2 are similar to Theorem 3.2 where the equilibrium $\overline{E}$ has been replaced by $\overline{E}_c$.

*Remark* 4.1. The effect of diffusion in the case of the cooperation model can be studied in a way similar to that of the competition model given in Section 3. It may be noted here that the results of Theorem 3.3 are also valid in the case of cooperation.

## 5. PREY–PREDATOR MODEL

We consider $x_1$ and $x_2$ to be prey and predator respectively. Then in this case we have

$$r_{10} > 0,\ r_{20} < 0,\ a_{12} > 0, \text{ and } a_{21} < 0.$$

We take $a_{21} = -b_{21}$ and $r_{20} = -r'_{20}$, where $b_{21} > 0$, $r'_{20} > 0$.

In this case there exist three nonnegative equilibria, namely,

$$E_0\left(0,0,\frac{Q_0}{\delta_0(1-\theta_0\theta_1-\theta_0'\theta_2)},\frac{\theta_0 Q_0}{\delta_1(1-\theta_0\theta_1-\theta_0'\theta_2)},\right.$$
$$\left.\frac{\theta_0' Q_0}{\delta_2(1-\theta_0\theta_1-\theta_0'\theta_2)}\right),$$

$$\tilde{E}_p\left(\tilde{x}_{1p},0,\tilde{T}_p,\tilde{U}_{1p},\tilde{U}_{2p}\right),$$

and

$$\overline{E}_p\left(\bar{x}_{1p},\bar{x}_{2p},\overline{T}_p,\overline{U}_{1p},\overline{U}_{2p}\right).$$

$E_0$ exists if $1-\theta_0\theta_1-\theta_0'\theta_2>0$. The existence of $\tilde{E}_p$ and $\overline{E}_p$ can be established similarly to those of the competition model.

The local stability of $E_0$ and $\tilde{E}_p$ can be studied in a way similar to that used in Section 3 for the competition model.

The following theorem shows that $\overline{E}_p$ is locally asymptotically stable. The proof of this theorem is similar to Theorem 3.1 and hence is omitted.

THEOREM 5.1. *Let the inequalities*

$$\left\{\bar{k}_1\theta_1\delta_1+\bar{k}_2\left(\theta_0\delta_0+\lambda_1\bar{x}_{1p}\right)\right\}^2<\tfrac{1}{2}\bar{k}_1\bar{k}_2\delta_1\left(\delta_0+\lambda_1\bar{x}_{1p}+\lambda_2\bar{x}_{2p}\right)\quad(5.1\text{a})$$

$$\left\{\bar{k}_1\theta_2\delta_2+\bar{k}_3\left(\theta_0'\delta_0+\lambda_2\bar{x}_{2p}\right)\right\}^2<\tfrac{1}{2}\bar{k}_1\bar{k}_3\delta_2\left(\delta_0+\lambda_1\bar{x}_{1p}+\lambda_2\bar{x}_{2p}\right)\quad(5.1\text{b})$$

*hold*, *where*

$$\bar{k}_1=\min\left\{\frac{1}{4}\frac{a_{11}}{\lambda_1^2}\frac{\delta_0+\lambda_1\bar{x}_{1p}+\lambda_2\bar{x}_{2p}}{\overline{T}_p^2},\frac{1}{4}\frac{a_{22}}{\lambda_2^2}\frac{a_{12}}{b_{21}}\frac{\delta_0+\lambda_1\bar{x}_{1p}+\lambda_2\bar{x}_{2p}}{\overline{T}_p^2}\right\},$$
$$\bar{k}_2=\frac{r_{11}}{\lambda_1\overline{T}_p+\beta_1},$$
$$\bar{k}_3=\frac{a_{12}}{b_{21}}\frac{r_{21}}{\lambda_2\overline{T}_p+\beta_2},$$

*then $\overline{E}_p$ is locally asymptotically stable*.

In order to show the global stability of $\overline{E}_p$, we need the following lemma whose proof is easy and hence is omitted.

LEMMA 5.1. *The set*

$$\Omega_3 = \left\{ (x_1, x_2, T, U_1, U_2) : 0 \le x_1 \le \frac{r_{10}}{a_{11}}, 0 \le x_2 \le \frac{r_{10} b_{21}}{a_{11} a_{22}}, \right.$$
$$\left. 0 \le T + U_1 + U_2 \le L_3 \right\}$$

*attracts all solutions initiating in the interior of the positive orthant, where*

$$L_3 = \frac{1}{\delta} \frac{r_{10}}{a_{11}} \left( \beta_1 + \frac{b_{21}}{a_{22}} \beta_2 \right),$$
$$\delta = \min\{ \delta_0 (1 - \theta_0 - \theta_0'), \delta_1 (1 - \theta_1), \delta_2 (1 - \theta_2) \}.$$

In the following theorem we are able to write down conditions for $\overline{E}_p$ to be globally asymptotically stable. The proof of this theorem is similar to that of Theorem 3.2 and hence is omitted.

THEOREM 5.2. *Let the inequalities*

$$\left\{ \hat{k}_1 \theta_1 \delta_1 + \hat{k}_2 (\theta_0 \delta_0 + \lambda_1 \bar{x}_{1p}) \right\}^2 < \tfrac{1}{2} \hat{k}_1 \hat{k}_2 \delta_1 (\delta_0 + \lambda_1 \bar{x}_{1p} + \lambda_2 \bar{x}_{2p}) \quad (5.2a)$$

$$\left\{ \hat{k}_1 \theta_2 \delta_2 + \hat{k}_3 (\theta_0' \delta_0 + \lambda_2 \bar{x}_{2p}) \right\}^2 < \tfrac{1}{2} \hat{k}_1 \hat{k}_3 \delta_2 (\delta_0 + \lambda_1 \bar{x}_{1p} + \lambda_2 \bar{x}_{2p}) \quad (5.2b)$$

*hold, where*

$$\hat{k}_1 = \min \left\{ \frac{1}{4} \frac{a_{11}}{\lambda_1^2} \frac{\delta_0 + \lambda_1 \bar{x}_{1p} + \lambda_2 \bar{x}_{2p}}{L_3^2}, \frac{1}{4} \frac{a_{22}}{\lambda_2^2} \frac{a_{12}}{b_{21}} \frac{\delta_0 + \lambda_1 \bar{x}_{1p} + \lambda_2 \bar{x}_{2p}}{L_3^2} \right\},$$
$$\hat{k}_2 = \frac{r_{11}}{\lambda_1 L_3 + \beta_1},$$
$$\hat{k}_3 = \frac{a_{12} r_{21}}{b_{21} (\lambda_2 L_3 + \beta_2)},$$

*then $\overline{E}_p$ is globally asymptotically stable with respect to all solutions initiating in the interior of the positive orthant.*

*Remark* 5.1. The effect of diffusion in the case of the prey–predator model is found to be similar to the competition model given in Section 3. In particular, the results of Theorem 3.3 remain valid in this case.

## 6. NUMERICAL EXAMPLES

In this section we present numerical examples to explain the applicability of the results discussed above. We choose the following values of the parameters in model (2.1) without diffusion

$$\begin{gathered} r_{11} = 0.05, \quad r_{21} = 0.04, \quad a_{11} = 0.22, \quad a_{22} = 0.26, \\ Q_0 = 15.0 \quad \delta_0 = 6.7, \quad \delta_1 = 15.5, \quad \delta_2 = 10.4, \\ \theta_1 = 0.02, \quad \theta_2 = 0.03, \quad \theta_0 = 0.01, \quad \theta_0' = 0.04, \\ \lambda_1 = 0.06, \quad \lambda_2 = 0.09, \quad \beta_1 = 0.25, \quad \text{and} \quad \beta_2 = 0.3. \end{gathered} \tag{6.1}$$

EXAMPLE 1. In this example we consider the case when the two species compete with each other. In addition to the values of the parameters given in Eq. (6.1), we choose the following parameters in model (2.1) without diffusion:

$$r_{10} = 5.0, r_{20} = 3.0\ a_{12} = 0.07, \text{and } a_{21} = 0.08.$$

With the above values of the parameters, it can be checked that the interior equilibrium $\overline{E}$ exists and is given by

$$\bar{x}_1 = 21.01420, \bar{x}_2 = 5.03089, \overline{T} = 1.81106, \overline{U}_1 = 0.49409, \overline{U}_2 = 0.27064.$$

It can also be checked that conditions (3.8) in Theorem 3.1 are satisfied which shows that $\overline{E}$ is locally asymptotically stable.

Further, we note that the conditions (3.9) in Theorem 3.2 are also satisfied which shows that $\overline{E}$ is globally asymptotically stable.

EXAMPLE 2. Here we consider the case when the two species cooperate with each other. In addition to the values of the parameters given in Eq. (6.1), we choose the following values of the parameters in model (2.1) without diffusion:

$$r_{10} = 5.0, r_{20} = 3.0, a_{12} = -0.07, \text{and } a_{21} = -0.08.$$

With the above values of the parameters, it can be verified that the interior equilibrium $\overline{E}_c$ exists and is given by

$$\bar{x}_{1c} = 29.05284, \bar{x}_{2c} = 20.34072, \overline{T}_c = 1.50652,$$

$$\overline{U}_{1c} = 0.64453, \overline{U}_{2c} = 0.89076.$$

It can also be verified that conditions (4.1) in Theorem 4.1 are satisfied, showing the local stability character of $\overline{E}_c$.

Further, it is easy to verify that conditions (4.2) in Theorem 4.2 are satisfied, showing the global stability character of $\overline{E}_c$.

EXAMPLE 3. In this example we consider the case when $x_2$ preys on $x_1$. In addition to the values of the parameters given in Eq. (6.1), we choose the following values of the parameters in model (2.1) without diffusion:

$$r_{10} = 5.0, r_{20} = -0.5, a_{12} = 0.2, \text{ and } a_{21} = -0.1.$$

With the above values of the parameters, it can be verified that the interior equilibrium $\overline{E}_p$ exists, and is given by

$$\bar{x}_{1p} = 18.09059, \bar{x}_{2p} = 4.99304, \overline{T}_p = 1.84798, \overline{U}_{1p} = 0.42918, \overline{U}_{2p} = 0.27150.$$

It can also be verified that the conditions (5.1) in Theorem 5.1 are satisfied. This shows that $\overline{E}_p$ is locally asymptotically stable.

Further, it can also be checked that the conditions (5.2) in Theorem 5.2 are satisfied. This shows that $\overline{E}_p$ is globally asymptotically stable.

## 7. SUMMARY

In this paper we have proposed and analysed a mathematical model for studying the survival of two interacting species in a polluted environment, the modes of interaction being competition, cooperation, and predation. The model has been analysed with and without diffusion. When there is no diffusion it has been shown that in the case of a constant introduction of pollutant into the environment the competing species settle down to their respective equilibrium levels, the magnitude of which depends upon the equilibrium levels of washout and uptake rates of the pollutant. It has also been noted that if the concentration of pollutant increases unabatedly, then the survival of the species would be threatened. In the case of an instantaneous introduction of the pollutant into the environment, it has been found that the competing species again settle down to their respective equilibrium levels whose magnitude is higher than in the case of a constant introduction of pollutant into the environment.

The effect of diffusion on the interior equilibrium state of the system has also been investigated. It has been shown that if the positive equilibrium of the system without diffusion is globally asymptotically stable, then the corresponding uniform steady state of the system with diffusion is also globally asymptotically stable. It has further been noted that if the positive equilibrium of the system with no diffusion is unstable, then the corresponding uniform steady state of the system with diffusion can be made stable by increasing the diffusion coefficients appropriately. From the proof of Theorem 3.3, it should be noted that $\dot{U}$ contains some extra negative terms which imply that the global stability is more feasible in the case of diffusion than in the case of no diffusion. In the cases of cooperation and prey–predator, similar results have been found. In each case, a numerical example has been given to illustrate the results obtained.

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